2023 June 1

2) Let $\alpha \in \mathcal{C}(\mathbb{R})$ be a continuously differeniable function satisfying $\alpha(0) = 0$ and $\alpha'(x) > 0$ at each $x \in \mathbb{R}$. For each $j \in \mathbb{N}$, define the function $f_j, g_j \in \mathcal{C}([0,1])$ by

$$f_j(x) = x + \frac{1}{j}, g_j(x) = \frac{j\alpha(x)}{1 + j\alpha(x)}$$

Evaluate $\lim_{j\to\infty} \int_0^1 f_j(x) dg_j(x)$.

Proof. First we compute g_j',

$$g_j' = \frac{j\alpha'(1 + j\alpha) - j\alpha' j\alpha}{(1 + j\alpha)^2} = \frac{j\alpha'}{(1 + j\alpha)^2} > 0$$

The last inequality is because $\alpha' > 0$. So we know that g_j is monotone increasing.

Note α is continuous so $j/(1 + j\alpha)^2$ is continuous. And by theorem 6.8, continuous functions are integrable, so $\alpha' \in \mathcal{R}$ and $j/(1 + j\alpha)^2 \in \mathcal{R}$. Then by theorem 6.12, $g_j' = j\alpha'/(1 + j\alpha)^2 \in \mathcal{R}$.

Note f_j is obviously bounded on $[0,1]$. Now by theorem 6.17, we have $\int_0^1 f_j dg_j = \int_0^1 f_j g_j' dx$.

We would apply integration by parts here to $\int_0^1 f_j g_j' dx$, and we have

$$\int_0^1 f_j g_j' dx = f_j(1)g_j'(1) - f_j(0)g_j'(0) - \int_0^1 f_j' g_j dx = (1 + \frac{1}{j})\frac{j\alpha(1)}{1 + j\alpha(1)} - 0 - \int_0^1 \frac{j\alpha(x)}{1 + j\alpha(x)} dx$$

So the question wants

$$\lim_{j\to\infty} (1 + \frac{1}{j})\frac{j\alpha(1)}{1 + j\alpha(1)} - \lim_{j\to\infty} \int_0^1 \frac{j\alpha(x)}{1 + j\alpha(x)} dx$$

Let's evaluate the second term first. We'll show $\lim_{j\to\infty} \int_0^1 \frac{j\alpha(x)}{1+j\alpha(x)} dx = 1$. For any $\epsilon > 0$. We want to show

$$\lim_{j\to\infty} \int_0^\epsilon \frac{j\alpha(x)}{1 + j\alpha(x)} dx + \lim_{j\to\infty} \int_\epsilon^1 \frac{j\alpha(x)}{1 + j\alpha(x)} dx = 1$$

We can separate the interval of integration because $\frac{j\alpha(x)}{1+j\alpha(x)}$ is continous and therefore integrable, and we applied theorem 6.12. Note since $\alpha(0) = 0, \alpha' > 0$, we know that $0 \le \alpha(x), x \in [0, \epsilon]$, so $0 \le \frac{j\alpha(x)}{1+j\alpha(x)} \le 1$. By theorem 6.12 again, we have $0 \le \int_0^\epsilon \frac{j\alpha(x)}{1+j\alpha(x)} dx \le \epsilon$, so

$$|\lim_{j\to\infty} \int_0^\epsilon \frac{j\alpha(x)}{1 + j\alpha(x)} dx| = \lim_{j\to\infty} \int_0^\epsilon \frac{j\alpha(x)}{1 + j\alpha(x)} dx < \epsilon$$

Now for $\lim_{j\to\infty}\int_\epsilon^1 \frac{j\alpha(x)}{1+j\alpha(x)}dx$. We've stablished $\frac{j\alpha(x)}{1+j\alpha(x)} \in \mathcal{R}$. Now we show $\{\frac{j\alpha(x)}{1+j\alpha(x)}\}_j$ uniformly converges to 1. Fix $\epsilon' > 0$.

$$|\frac{j\alpha(x)}{1+j\alpha(x)} - 1| = |\frac{-1}{1+j\alpha(x)}|$$

Note $\forall x \in [\epsilon, 1], 0 < \alpha(\epsilon) \le \alpha(x) \le M$, where $|\alpha| \le M$ because α is continuous on a compact set and therefore bounded. So we know $\alpha(x)$ is non-zero and bounded on $[\epsilon, 1]$. So there is a j big enough such that $|\frac{-1}{1+j\alpha(x)}| < \epsilon' \forall x \in [\epsilon, 1]$, so $\{\frac{j\alpha(x)}{1+j\alpha(x)}\}_j \to 1$ uniformly. Now we apply theorem 7.14, and we get

$$\lim_{j\to\infty}\int_\epsilon^1 \frac{j\alpha(x)}{1+j\alpha(x)}dx = \int_\epsilon^1 \lim_{j\to\infty}\frac{j\alpha(x)}{1+j\alpha(x)}dx = \int_\epsilon^1 1dx = 1-\epsilon$$

So we have

$$|\lim_{j\to\infty}\int_0^\epsilon \frac{j\alpha(x)}{1+j\alpha(x)}dx + \lim_{j\to\infty}\int_\epsilon^1 \frac{j\alpha(x)}{1+j\alpha(x)}dx - 1| \le \epsilon + (1-\epsilon) - 1 = 0$$

Since the absolute value of the difference of $\lim_{j\to\infty}\int_0^\epsilon \frac{j\alpha(x)}{1+j\alpha(x)}dx + \lim_{j\to\infty}\int_\epsilon^1 \frac{j\alpha(x)}{1+j\alpha(x)}dx$ and 1 is less than or equal to 0, the two things must be equal.

Now we go on to compute $\lim_{j\to\infty}(1+\frac{1}{j})\frac{j\alpha(1)}{1+j\alpha(1)}$.

$$\lim_{j\to\infty}(1+\frac{1}{j})\frac{j\alpha(1)}{1+j\alpha(1)} = \lim_{j\to\infty}\frac{\alpha(1)}{1/j+\alpha(1)} + \lim_{j\to\infty}\frac{\alpha(1)}{1+j\alpha(1)} = 1 + 0 = 1$$

Collecting all our evidence, we see that

$$\lim_{j\to\infty}\int_0^1 f_j(x)dg_j(x) = \lim_{j\to\infty}\int_0^1 f_j g_j' dx = \lim_{j\to\infty}(1+\frac{1}{j})\frac{j\alpha(1)}{1+j\alpha(1)} - \lim_{j\to\infty}\int_0^1 \frac{j\alpha(x)}{1+j\alpha(x)}dx = 1 - 1 = 0$$

So the answer is 0. □

2023 Jan 18

1) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in \mathbb{R}$. Let $\{a_n\}_{n=1}^{\infty} \subset (-\infty, x_0)$ and $\{b_n\}_{n=1}^{\infty} \subset (x_0, -\infty)$ be sequences that both converge to x_0. Prove that

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(x_0)$$

(You may assume that $x_0 = 0$).

Proof. By definition, we'll show that for any $\epsilon > 0$, there exists an N such that $n > N$ implies $|\frac{f(b_n)-f(a_n)}{b_n-a_n} - f'(0)| < \epsilon$. Consider the following steps:

$$\begin{aligned}
\frac{f(b_n) - f(a_n)}{b_n - a_n} &= \frac{f(b_n) - f(0)}{b_n - a_n} - \frac{f(a_n) - f(0)}{b_n - a_n} \\
&= \frac{b_n}{b_n - a_n}\frac{f(b_n) - f(0)}{b_n} - \frac{a_n}{b_n - a_n}\frac{f(a_n) - f(0)}{a_n} \\
&= \frac{b_n}{b_n - a_n}\frac{f(b_n) - f(0)}{b_n} + \frac{|a_n|}{b_n - a_n}\frac{f(a_n) - f(0)}{a_n}
\end{aligned}$$

The last equality is because $a_n < 0$. Since $f'(0)$ exists, we know that $\lim_{t\to 0} \frac{f(t)-f(0)}{t} = f'(0)$. And since $b_n \to 0, a_n \to 0$, by theorem 4.2, we know that $\frac{f(b_n)-f(0)}{b_n} = f'(0)$ and $\frac{f(a_n)-f(0)}{a_n} = f'(0)$. For ϵ, we know there's a large enough N such that

$$|\frac{f(b_n) - f(0)}{b_n} - f'(0)| < \epsilon, |\frac{f(a_n) - f(0)}{a_n} - f'(0)| < \epsilon$$

Now note that $b_n - a_n = b_n + |a_n|$. So $\frac{b_n}{b_n-a_n} + \frac{|a_n|}{b_n-a_n} = 1$. So we have

$$\begin{aligned}
|\frac{f(b_n) - f(a_n)}{b_n - a_n} - f'(0)| &= |\frac{b_n}{b_n - a_n}\frac{f(b_n) - f(0)}{b_n} + \frac{|a_n|}{b_n - a_n}\frac{f(a_n) - f(0)}{a_n} - \frac{b_n}{b_n - a_n}f'(0) - \frac{|a_n|}{b_n - a_n}f'(0)| \\
&= |\frac{b_n}{b_n - a_n}(\frac{f(b_n) - f(0)}{b_n} - f'(0)) + \frac{|a_n|}{b_n - a_n}(\frac{f(a_n) - f(0)}{a_n} - f'(0))| \\
&\le |\frac{b_n}{b_n - a_n}(\frac{f(b_n) - f(0)}{b_n} - f'(0))| + |\frac{|a_n|}{b_n - a_n}(\frac{f(a_n) - f(0)}{a_n} - f'(0))| \\
&= \frac{b_n}{b_n - a_n}|(\frac{f(b_n) - f(0)}{b_n} - f'(0))| + \frac{|a_n|}{b_n - a_n}|(\frac{f(a_n) - f(0)}{a_n} - f'(0))| \\
&< \frac{b_n}{b_n - a_n}\epsilon + \frac{|a_n|}{b_n - a_n}\epsilon \\
&= \epsilon
\end{aligned}$$

as desired. □

2) ✓ Let $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \subset (0,\infty)$ be given.

(a) Assume that $\limsup_{n\to\infty}(\frac{a_n}{b_n}) < \infty$. Prove there exists $M \in \mathbb{R}$ such that $a_n \le M b_n$, for all $n \in \mathbb{N}$.

Proof. Let $\limsup_{n\to\infty}(\frac{a_n}{b_n}) = L$. By definition of lim sup, $\lim_{n\to\infty}(\sup\{a_k/b_k | k \ge n\}) = L$. By the definition of limit, exists N such that $n \ge N$ implies $\sup\{a_k/b_k|k \ge n\} \le 2L$, which means for $n \ge N$, $a_k \le 2Lb_k$. Note $a_m \le \frac{a_m}{b_m} b_m$. Let $M = 2L \vee \max\{\frac{a_m}{b_m} | m < N\}$. □

(b) Suppose the sequence $\{\frac{a_n}{b_n}\}_{n=1}^\infty$ converges in $\mathbb{R}$. Must $\limsup_{n\to\infty}(a_n/b_n)$ be finite?

Proof. The sequence converges meaning $\lim\{\frac{a_n}{b_n}\} = L < \infty$ exists. Remark given in class says $\lim\{\frac{a_n}{b_n}\} = L$ iff $\limsup\{\frac{a_n}{b_n}\} = \liminf\{\frac{a_n}{b_n}\} = L$. So $\limsup\{\frac{a_n}{b_n}\}$ must be finite also. □

3) a) Suppose that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence from $(C([0,1]), \rho_\infty)$. Determine whether $\{f_n\}_{n=1}^\infty$ must be uniformly equicontinuous.

Proof. Since $\{f_n\}_{n=1}^\infty$ is Cauchy, it is convergent in $(C([0,1]), \rho_\infty)$ by theorem 7.8. By theorem 7.24, $\{f_n\}_{n=1}^\infty$ is equicontinuous. □

b) Suppose that $F \subset (C([0,1]), \rho_\infty)$ is closed and bounded but not uniformly equicontinuous. Prove that F is not compact in $(C([0,1]), \rho_\infty)$.

Proof. AFSOC F is compact. We'll show that F is equicontinuous.

To prove equicontinuity, fix $\epsilon > 0$. We want to show that there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|fx - fy| < \epsilon$ for all $f \in F$.

$\{N_{\epsilon/3}(f) | f \in F\}$ is an open cover of F. By compactness, there's a finite subcover, say centered around $f_1, ..., f_k$. Note each f_i is uniform continuous because the domain is compact. So for $\epsilon/3$ there is δ_i such that $|x - y| < \delta_i$ imples $|f_i x - f_i y| < \epsilon/3$. Since we only have finite f_i's, we can take $\delta = \min_i \delta_i$.

Fix some $f \in F$, there is some f_i such that $\forall x |fx - f_i x| < \epsilon/3$. So let $|x - y| < \delta$, we have

$$|fx - fy| \le |fx - f_i x| + |f_i x - f_i y| + |f_i y - fy| < \epsilon$$

□

4) Question 4 is wrong as written, confirmed by author. New statement suggested by coauthor:

Use the Riemann condition to show that $f \in \mathcal{R}_\alpha[0,2]$ where $f(x) = \frac{\pi x}{8}$ and

$$\alpha(x) = \begin{cases} x+1 & 0 \le x \le 1 \\ 4x & 1 < x \le 2 \end{cases}$$

Compute the value of the Riemann-Stieltjes integral $\int_0^2 f(x)d\alpha$.

Proof. The Riemann condition is Theorem 6.6 in the book: $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if for every $\epsilon > 0$ there exists a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. Note f is uniformly continuous because it's continuous on a compact interval. So for $\epsilon/5$, there's a $\delta > 0$ such that $|x-y| < \delta$ imples $|f(x) - f(y)| < \epsilon/2$. For this problem, we choose a partition P such that $1 = x_j$ for some j, and for every i, $\Delta x_i < \delta$. So we get

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{n=1}^{n}(M_i - m_i)\Delta\alpha_i \\ &= \sum_{n=1}^{j}(M_i - m_i)(x_i + 1 - x_{i-1} - 1) + \sum_{n=j+1}^{n}(M_i - m_i)4(x_i - x_{i-1}) \\ &\le \epsilon/5\sum_{n=1}^{j}(x_i - x_{i-1}) + \epsilon/5\sum_{n=j+1}^{n}(x_i - x_{i-1}) \\ &= \frac{\epsilon}{5}\cdot(1-0) + 4\frac{\epsilon}{5}\epsilon\cdot(2-1) \\ &= \epsilon \end{aligned}$$

So we can make $U(P,f,\alpha) - L(P,f,\alpha)$ arbitrary close together, so by the riemann condition, f is riemann-stieltjes integrable.

By Theorem 6.12, $\int_0^2 f d\alpha = \int_0^1 f d\alpha + \int_1^2 f d\alpha$.

By Theorem 6.12 $\int_0^1 f d\alpha = \int_0^1 f d(x+1) = \int_0^1 f dx + \int_0^1 f d(1) = \int_0^1 \frac{\pi}{8} x dx + 0 = \frac{\pi}{8}\frac{1}{2} = \frac{\pi}{16}$.

By Theorem 6.17, $\int_1^2 f d\alpha = \int_1^2 f d(4x) = \int_1^2 f(4x)' dx = \int_1^2 4f dx = \int_1^2 \frac{\pi}{2} x dx = \frac{\pi}{2}\frac{3}{2} = \frac{3\pi}{4}$.

So $\int_0^2 f d\alpha = \frac{\pi}{16} + \frac{3\pi}{4}$. □

5) Determine all the values of $x \in \mathbb{R}$ for which the series below converges

$$\sum_{n=1}^{\infty} \frac{x^n}{1+n|x|^n}$$

.

Proof. The series converges when $x < 1$ and diverges when $x \geq 1$. Let $|x| < 1$. So we know $\frac{1}{|x|}^n = (1+p)^n$ for some $p > 0$. Let $|a_n| = |\frac{x^n}{1+n|x|^n}|$ and $b_n = \frac{1}{n^2}$. So we have

$$\begin{aligned}
\frac{|a_n|}{b_n} = \frac{n^2|x|^n}{|1+n|x|^n|} &= \frac{n^2|x|^n}{1+n|x|^n} = \frac{n^2}{\frac{1}{|x|^n}+n} \\
&= \frac{n^2}{(1+p)^n+n} \\
&= \frac{1}{\frac{(1+p)^n}{n^2}+\frac{1}{n}} \\
&= \frac{1}{\frac{(1+p)^n}{n^2}+\frac{1}{n}} \\
&= \frac{1}{\frac{(1+p)^n}{n^2}+0} \\
&= \frac{n^2}{(1+p)^n}
\end{aligned}$$

Note that $\frac{n^2}{(1+p)^n} = 0$ by theorem 3.20 d). The second equality is because $1+n|x|^n$ is positive anyway. The fourth equality is bcause $\frac{1}{|x|}^n = (1+p)^n$. The third last equality is because of limit rules and $\frac{(1+p)^n}{n^2} \neq 0$. In conclusion, we have $\frac{|a_n|}{b_n} = 0$, this means that $|a_n| < b_n = \frac{1}{n^2}$ for all large enough n. By Weierstrass M-test, we know that the series converges for $|x| < 1$.

When $x \leq -1$, we rearrange to the series $\sum \frac{(-1)^n}{\frac{1}{|x|^n}+n}$ is alternating. Note $0 \leq \frac{1}{|x|}^n \leq 1$, so $\frac{1}{|x|^n}+n \to \infty$, so $\frac{1}{\frac{1}{|x|^n}+n} \to 0$. And note $\frac{1}{\frac{1}{|x|^n}+n}$ is decreasing. So by theorem 3.43, the series converges.

Let $x \geq 1$. So $0 < \frac{1}{x^n} \leq 1$.

$$\frac{x^n}{1+n|x|^n} = \frac{1}{\frac{1}{|x|^n}+n} \geq \frac{1}{1+n} \geq \frac{1}{2n}$$

The first equality is because $|xy| = |x||y|$ by theorem 1.33 c) and $1+n|x|^n$ is positive. The first inequality is because $\frac{1}{|x|^n} \leq 1$. The last is because $n \geq 1$. Since $\frac{1}{2}\sum\frac{1}{n}$ diverges, we know that the series also diverges.

□

2022 June 1

3) Consider the sequence $\{x_n\}_{n\geq 1}$ defined by $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n = 1, 2, \ldots$. Show that $x_n \to 0$ as $n \to \infty$. Also, show that $\frac{x_{n+1}}{x_n} \to \frac{1}{2}$.

Proof. We'll show the sequences is monotone decreasing and bounded. But first we show $0 < (1 - x_n) < 1$ by induction.

BC: $0 < x_1 < 1$, so obviously $0 < (1 - x_1) < 1$.

IH: suppose $0 < (1 - x_{n-1}) < 1$

IS: taking the square root, we get $0 < \sqrt{(1 - x_{n-1})} < 1$. By simple arithmetic, we get $1 > 1 - \sqrt{(1 - x_{n-1})} > 0$. So $0 < x_n < 1$, implying $0 < (1 - x_n) < 1$. Since $0 < (1 - x_n) < 1$, we know

$$\begin{aligned}(1 - x_n)^2 &< (1 - x_n)\\ (1 - x_n) &< \sqrt{(1 - x_n)}\\ x_n &> 1 - \sqrt{(1 - x_n)}\\ x_n &> x_{n+1}\end{aligned}$$

So the sequence is monotonic decresing, and note the terms are bounded below by 0, so by theorem 3.14, the sequence is convergent. So we can write, exists some L such that

$$\lim_{n\to\infty} x_n = L = \lim_{n\to\infty} x_{n+1}$$

So we get

$$\begin{aligned}x_{n+1} &= 1 - \sqrt{1 - x_n}\\ L &= 1 - \sqrt{1 - L}\\ (1 - L)^2 &= (1 - L)\end{aligned}$$

The only solutions are $1 - L = 0$ or $1 - L = 1$, yielding $L = 0$ or $L = 1$. $L = 1$ is not the limit because the sequence is strictly monotone decresing and $x_1 < 1$ for all n. So $L = 0$ has got to be the limit of the sequence. So we've shown $x_n \to 0$.

Now we compute $\frac{x_{n+1}}{x_n}$.

$$\begin{aligned}\frac{x_{n+1}}{x_n} &= \frac{1 - \sqrt{1 - x_n}}{x_n} = \frac{(1 - \sqrt{1 - x_n})(1 + \sqrt{1 - x_n})}{x_n(1 + \sqrt{1 - x_n})}\\ &= \frac{(1 - 1 + x_n)}{x_n(1 + \sqrt{1 - x_n})} = \frac{1}{(1 + \sqrt{1 - x_n})}\\ &= \frac{1}{(1 + \sqrt{1 - x_n})} = \frac{1}{2}\end{aligned}$$

as desired.

□

5) Find the domain of convergence and the sum of the series

$$\sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{2n+1}$$

Show how one may use the sum of the series to provide an approximation for π up to three decimals. Be sure to provide all technical details.

Proof. The ratio test gives that the series converges if $\limsup |\frac{(-1)^{n+1}x^{2n+3}}{2n+3}\frac{2n+1}{(-1)^n x^{2n+1}}| < 1$. Note

$$\limsup |\frac{(-1)^{n+1}x^{2n+3}}{2n+3}\frac{2n+1}{(-1)^n x^{2n+1}}| = x^2 \limsup |\frac{2n+1}{2n+3}| = x^2$$

So the series would converge if $|x| < 1$. And it would diverge if $|x| > 1$.

The series also converges at $|x| = 1$. $\sum_{n\geq 0}(-1)^n \frac{1}{2n+1}$ has the properties $|\frac{1}{2n+1}| \to 0$ and monotone decresing, so by theorem 3.43, the series converges.

This is the taylor expansion of $\arctan(x)$, so the sum of the series

$$\sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{2n+1} = \arctan(x)$$

To approximate π with the series which has radius of convergence of 1, we need to plug in some $x \in [-1, 1]$. By the unit circle, we know that $\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$. So we can plug in $\frac{1}{\sqrt{3}}$ to the series, evaluate the first few terms and multiply by 6. The number of terms to evaluate is given by: let $\epsilon = 0.0001$, since the series is convergent, there's an N such that $\sum_{n\geq N}(-1)^n \frac{x^{2n+1}}{2n+1} < \epsilon$. N is the number of terms you should sum. □

2022 Jan 12

1) Suppose $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f' : [a, b] \to \mathbb{R}$ is continuous. Show that $\forall \epsilon > 0, \exists \delta > 0$ such that for every $x, y \in [a, b]$ with $|x - y| < \delta$, we have

$$|\frac{f(x) - f(y)}{x - y} - f'(x)| < \epsilon$$

Proof. Fix $\epsilon > 0$. Since $f'(x)$ continuous, we know that there's a ⁏ > 0 such that if $|p - x| <$ ⁏$then$|f'(p)-f'(x)|<ϵ. Let =⁏. So we have $|x - y| <$ ⁏$.Bymeanvaluetheorem, wehave \frac{f(x)-f(y)}{x-y} = f'(\xi)for\xi \in (x, y)$, so we also have $|\xi - x| <$ ⁏, so we have

$$|f'(\xi) - f'(x)| = |\frac{f(x) - f(y)}{x - y} - f'(x)| < \epsilon$$

as desired.

□

3) Compute, with proof $\lim_{k\to\infty} \sum_{n=1}^{\infty} n^{-k}$

Proof. We'll show that $\lim_{k\to\infty} \sum_{n=1}^{\infty} n^{-k} = 1$, which is equivalent to showing

$$\lim_{k\to\infty} \sum_{n=2}^{\infty} n^{-k} = 0$$

Note that we know $\sum_{n=2} \frac{1}{n^2} = L$ for some $L \in \mathbb{R}$ because it is a convergent series.

Note for $\sum_{n=2} \frac{1}{n^{2+k}}$, we have

$$\sum_{n=2} \frac{1}{n^{2+k}} = \sum_{n=2} \frac{1}{n^2} \frac{1}{n^k} \le \sum_{n=2} \frac{1}{n^2} \frac{1}{2^k} = \frac{1}{2^k} L$$

Let k go to infinity we have

$$\lim_{k\to\infty} \sum_{n=2}^{\infty} n^{-k} = \lim_{k\to\infty} \sum_{n=2} \frac{1}{n^{2+k}} = \lim_{k\to\infty} \frac{1}{2^k} L = L \lim_{k\to\infty} \frac{1}{2^k} = L \cdot 0 = 0$$

as desired.

□

4) a) Let $f : [1,2] \to \mathbb{R}$ be a continuous function. If $\int_1^2 x^{-n} f(x)dx = 0$ for all integers $n \geq 0$, show that $f = 0$.

Proof. We'll apply the general Stone-Weierstrass theorem here: Let X be a compact metric space. $C_R(X)$ be the set of continuous function from X into the reals. Let $A \subset C_R(X)$ be a subspace satisfying $1 \in A$, $\forall a, b \in A, ab \in A$ and $\forall x, y \in X, \exists a \in A, a(x) \neq a(y)$ then A is dense in $C_R(X)$. Here we let $A = \{x^{-n} | n \in \mathbb{N}_0\}$. Note $x^0 = 1 \in A$. $x^{-j}x^{-k} = x^{-(j+k)} \in A$. $\frac{1}{x}$ would give different answers for different input because it's monotone on $[1,2]$. So we have that A is dense in $C_R(x)$.

Since $f \in C_R(x)$ and A is dense in it, we know that f is the uniform limit of $\{P_n\}$ where $P_n = a_0 + a_1 x^{-1} + a_2 x^{-2} + ... + a_k x^{-k}$.

We'll show $\int_1^2 f^2 = 0$.

$$\begin{aligned}
\int_1^2 f^2 &= \int_1^2 f(P_n) \\
&= \int_1^2 f P_n \\
&= a_0 \int_1^2 f + a_1 \int_1^2 x^{-1} f + a_2 \int_1^2 x^{-2} f + ... + a_k \int_1^2 x^{-k} f \\
&= 0 + 0 + ... + 0 \\
&= 0
\end{aligned}$$

The first equality is an application of theorem 7.14 because all P_n's are integrable on $[1,2]$ because they're continuous there. Second equality is theorem 6.12, third equality is by assumption. By exercise 2 in chapter 6, we know that $f^2 = 0$, which implies $f = 0$. □

b) Let $g; [1,2] \to \mathbb{R}$ be a differentiable function such that $g' : [1,2] \to \mathbb{R}$ is continuous. If $\int_1^2 x^{-n} dg(x) = 0$ for all integers $n \geq 0$, show that g is constant.

Proof. Since g' is continuous, we know that $g \in BV$. So there're monotone increasing functions u, l such that $g = u - l$ and $g' = u' - l'$. So we have

$$\int_1^2 x^{-n} dg(x) = \int_1^2 x^{-n} d(u + l) = \int_1^2 x^{-n} du + \int_1^2 x^{-n} dl$$

Note since u, l monotone increasing and x continuous, by theorem 6.9, $u, l \in \mathcal{R}$. So by theorem 6.17, we can write

$$\int_1^2 x^{-n} du + \int_1^2 x^{-n} dl = \int_1^2 x^{-n} u' dx + \int_1^2 x^{-n} l' dx = \int_1^2 x^{-n} (u + l)' dx$$

By the assumption, we know $\int_1^2 x^{-n}(u + l)' dx = 0$ for all $n \geq 0$, then apply part a) we know $(u + l)' = g' = 0$. By calc 1, we know g is a constant. □

5) ✓ Prove the following special case of Dini's Theorem: if $(f_n : [0,1] \to \mathbb{R})_{n=1}^{\infty}$ is decreasing sequence oof continuous functions such that $\lim_{n\to\infty} f_n(x) = 0$ for all $\in [0,1]$, then $(f_n)_{n=1}^{\infty}$ converges uniformly to 0. (You should not use any form of Dini's theorem without proof.)

Proof. (1) By definition of uniform convergence, we'll show $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N$ implies $\forall x \in [0,1], |f_n(x) - 0| < \epsilon$. Fix some $\epsilon > 0$. For each n, define set $K_n = \{x \in [0,1] : |f_n(x)| \geq \epsilon\}$. Note that $[-\epsilon, \epsilon]$ is a closed interval, and since f_n is continuous, we know that K_n is also closed. Since $K_n \subset [0,1]$ and $[0,1]$ compact, we know that K_n is also compact.

Now, since $f_n \geq f_{n+1}$, we have that $K_n \supset K_{n+1}$. Since for any $x \in [0,1]$, $f_n(x)$ converges to 0, by definition, there is some N such that $|f_N(x)| < \epsilon$, i.e. $x \notin K_N$. So $\bigcap K_n$ is empty.

We know by theorem 2.36 that nested sets of "non-empty" compact set have non-empty intersection. So by contrapositive, we must have that some K_N is empty, which also means all K_n where $n \geq N$ are empty since they're nested. This corresponds to for $n \geq N, \forall x \in [0,1], |f(x)| < \epsilon$, as desired. □

Proof. (2) Here we'll harness the finiteness of the subcovers of compact sets. Fix $\epsilon > 0$. Define $O_n = f_n^{-1}((-\infty, \epsilon)) = f_n^{-1}([0, \epsilon))$. The last equality is because $f_n(x) \geq 0$ for all x because $f_n(x)$ is a decreasing sequence that converges to 0. Since $(-\infty, \epsilon)$ open, and f continuous, we know that O_n is open. Since $f_n \geq f_{n+1}$, we also know that $O_n \subset O_{n+1}$. For each $x \in [0,1]$, since $f_n(x)$ converges to 0, exists some n such that $x \in O_n$. So the O_n's is an open cover of $[0,1]$. By compactness, there's a finite subcover. And since $O_n \subset O_{n+1}$, the one with the largest index N covers $[0,1]$. Like $[0,1] \subset O_N \supset [0,1]$, so $O_N = [0,1]$. And for any $n \geq N$, we know $[0,1] \subset O_n \subset O_N = [0,1]$, so $O_n = [0,1]$ also for any $n \geq N$. This means for $n \geq N$, $\forall x \in [0,1], |f_n(x)| < \epsilon$ by definition of the O_n's. We're done. □

6) Let $f : [0,1] \to \mathbb{R}$ be a continuous function

a) Show $\int_0^1 f(x^{\frac{1}{n}})dx = f(1)$

Proof. Recall Theorem 7.16 which states if on interval $[a,b]$, α is monotone increasing, $f_n \in \mathcal{R}(\alpha)\forall n$, $f_n \to f$ uniformly, then $\int_a^b f d\alpha = \int_a^b f_n d\alpha$.

Since $f, x^{1/n}$ continuous, their composition is continuous. And by theorem 6.8, $f(x^{1/n)} \in \mathcal{R}$

Since f and $x^{1/n}$ is continuous on $[0,1]$, the limit notation can move inside

$$f(x^{1/n}) = f(x^{1/n}) = f(x^0) = f(1)$$

By the theorem, we have $\int_0^1 f(x^{\frac{1}{n}})dx = \int_0^1 f(1)dx = f(1)$, as desired. □

b) If $f(x) > 0$ on $[0,1]$, show $\int_0^1 f(x)^{1/n}dx = 1$.

Proof. Again, similare to above. $f(x)^{1/n}$ is continuous on $[0,1]$. So

$$f(x)^{1/n} = f(x)^{1/n} = f(x)^0 = 1$$

By the theorem, we have $\int_0^1 f(x)^{\frac{1}{n}}dx = \int_0^1 1dx = 1$, as desired. □

2021 May 25

5) Suppose that $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty$ are sequences of strictly positive real numbers such that $\sum_{n=1}^\infty b_n$ converges, and suppose that for each integer $n \geq 1$, we have $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$. Show that $\sum_{n=1}^\infty a_n$ converges.

Proof. Rearranging $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$, we get $\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n}$. Let constant $C = \frac{a_1}{b_1}$. We know that for all n, $\frac{a_n}{b_n} \leq \frac{a_1}{b_1} = C$, so

$$|a_n| = a_n \leq C b_n$$

Note that $\sum_{n=1}^\infty C b_n = C \sum_{n=1}^\infty b_n$ which converges. By comparison test, $\sum_{n=1}^\infty a_n$ converges as well.

□

2021 Jan 19

1) Suppose $(X,d),(Y,\rho)$ are metric spaces; (Y,ρ) is compact; and $\phi : Y \to X$ is a continuous and onto function.

a) A well-know theorem states that if $F \subset Y$ is compact, then $\phi(F)$ is also compact. Prove this theorem, and conclude that (X,d) is a compact metric space.

Proof. See theorem 4.14. □

b) Suppose $G \subset X$ and $\phi^{-1}(G)$ is an open set. Prove that G is an open set.

Proof. It's equivalent to proving G^c is closed by theorem 2.23. Also by theorem 2.23, since $\phi^{-1}(G)$ is open, $(\phi^{-1}(G))^c$ is closed in the compact set Y. And by theorem 2.35, $(\phi^{-1}(G))^c$ is also compact. Since ϕ continuous, by theorem 4.14, $\phi((\phi^{-1}(G))^c)$ is compact, which is closed by theorem 2.34. Now we'll show $\phi((\phi^{-1}(G))^c) = G^c$.

We'll write out the set notation for each set.

$$\phi^{-1}(G) = \{y \in Y | \phi(y) \in G\}$$

$$(\phi^{-1}(G))^c = \{y \in Y | \phi(y) \notin G\}$$

$$\phi((\phi^{-1}(G))^c) = \{\phi(y) \in X | \phi(y) \notin G\} \subset G^c$$

To show the other containment, let $x \in G^c$, so $\phi^{-1}(x) \subset (\phi^{-1}(G))^c$. And $\{x\} = \phi(\phi^{-1}(x)) \subset \phi((\phi^{-1}(G))^c)$, as desired.

□

2) Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers. Be sure to include all details, prove that

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} \frac{a_1 + \ldots + a_n}{n}$$

Proof. Fix some $N \in \mathbb{N}$, for $n \ge N$ we have

$$\begin{aligned} \frac{a_1 + \ldots + a_n}{n} &= \frac{a_1 + \ldots + a_N}{n} + \frac{a_{N+1} + \ldots + a_n}{n} \\ &\ge \frac{a_1 + \ldots + a_N}{M} + \frac{n-N}{n} \inf\{a_n | n \ge N\} \\ &\ge \frac{a_1 + \ldots + a_N}{M} + \inf\{a_n | n \ge N\} \end{aligned}$$

In the second line, let $M \geq n$. Take $M \to \infty$, we get

$$\frac{a_1 + ... + a_n}{n} \geq \inf\{a_n | n \geq N\}$$

Take the $\inf_{n\geq N}$ of both sides we get

$$\inf_{n\geq N} \frac{a_1 + ... + a_n}{n} \geq \inf_{n\geq N} \inf\{a_n | n \geq N\} = \inf_{n\geq N} a_n$$

Then take $N \to \infty$, we get

$$\lim_{N\to\infty} \inf_{n\geq N} \frac{a_1 + ... + a_n}{n} \geq \lim_{N\to\infty} \inf_{n\geq N} a_n$$

as desired.

□

3) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $\int_a^b |f(t)| dt = 0$.

a) If f is continuous, prove that $f(t) = 0$ for every $t \in [a, b]$.

Proof. Proof 1 (black box proof):

Note since f continuous, $|f|$ is also continuous. Obviously $|f| \geq 0$, and we're given $\int_a^b |f| dx = 0$. By exercise 6.2, we know that $|f| = 0$, which implies $f = 0$.

Proof 2 (low level proof):

AFSOC, exsits $t \in [a, b]$ such that $|f(t)| > 0$. Since f is continous, $|f|$ continuoous. Fix $0 < \epsilon < |f(t)|$, there is a δ such that for all x with $|x - t| < \delta$, we have $||fx| - |ft|| < \epsilon$, i.e. $|f(x)| \geq |ft| - \epsilon > 0$.

Now let's examine $L(P, f)$ for P with $\Delta x_i < \delta$. Say $t \in [x_i, x_{i-1}]$.

$$\int_a^b f dx \geq L(P, f) = \sum_{k=1}^{n} \inf |f(x)| \Delta x_k \geq \inf |f(x)|(x_i - x_{i-1}) \geq (|f(t)| - \epsilon)(x_i - x_{i-1}) > 0$$

This is because since $\Delta x_i < \delta$ any $x \in [x_i, x_{i-1}]$ would be with in ϵ of $|f(t)|$. And we know $|f(x)| \geq |ft| - \epsilon > 0$. And $\Delta x_i > 0$ by definition.

□

b) Give an example (with proof) of a non-zero Riemann integrable function such that $\int_a^b |f| dx = 0$.

Proof. Define f to be 0 on $[0, 1)$ and f is 1 at $x = 1$. This is a non-zero function. And $\int_a^b f dx = \int_a^b |f| dx = 0$. Proof is trivial, omitted. □

5) Suppose (a_n) is a decreasing sequence of real numbers and $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} na_n = 0$.

Proof. Since $\sum_{n=1}^{\infty} a_n$ converges, by Theorem 3.23, $\lim_{n\to\infty} a_n = 0$. Since a_n's are decreasing, this implies $a_n \geq 0$.

Since $\sum_{n=1}^{\infty} a_n$ converges, there's an $N \in \mathbb{N}$ such that $\forall n, m \geq N$ we have $|\sum_{k=n}^{m} a_k| < \epsilon$ (Theorem 3.22). So for $n \geq N$, we have

$$|na_n| = na_n = a_n + ... + a_n \geq a_n + a_{n+1} + ... + a_{2n} = |\sum_{k=n}^{2n} a_k| < \epsilon$$

as desired.

□

6) For $x \in R$, consider the series, $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$.

a) Prove this series converges for every $x \in R$.

Proof. Note $\frac{1}{n^2+x^2} > 0$, so $|\frac{1}{n^2+x^2}| = \frac{1}{n^2+x^2} \le \frac{1}{n^2}$. We know $\sum \frac{1}{n^2}$ converges. By comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ also converges, independent of x. □

b) Set $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$. Prove that f is differentiable at each $x \in R$. Also, find a formula for $f'(x)$ (in terms of a series), being sure to justify that your formula is correct.

Proof. We'll use theorem 7.17 with $f_k(x) = \sum_{n=1}^{k} \frac{1}{n^2+x^2}$. Note f_k is differentiable, just take the derivative term by term. Note for $x_0 = 0$, $\{f_k(0) = \sum_{n=1}^{k} \frac{1}{n^2}\}$ converges by theorem 3.28. The last thing to show is that $\{f_k'(x) = -2\sum_{n=1}^{k} \frac{x}{(n^2+x^2)^2}\}$ converges uniformly. We'll use the M-test (Theorem 7.10) to show the derivative sequence converges uniformly. i.e. we'll show

$$\frac{|x|}{(n^2+x^2)^2} \le \frac{1}{n^2}$$

Equivalently, we'll show

$$|x|n^2 \le n^4 + x^4 + 2n^2x^2$$

If $|x| \ge 1$, then we know that $|x|n^2 \le n^2x^2 \le$RHS. If $|x| < 1$, then $|x|n^2 \le n^2 \le n^4 \le$ RHS because $n \ge 1$. So we can now apply theorem 7.17, we see that $f'(x) = \lim_{n\to\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{-2x}{(n^2+x^2)^2}$, showing f is differentiable at each $x \in R$ and giving a formula for it. □

2020 Jan 22

1) Let $\{a_n\}_{n=1}^{\infty} \subset (0, \infty)$ and $c > 0$ be given. Suppose that $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n$ diverges. Prove that there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} a_{n_k} = c$.

Proof. ✓ Since each a_n is positive, we know that $\{S_n\}$ the sequence of partial sums is monotonically increasing. And $\sum a_n$ must diverge to infinity instead of diverging in oscillatiion. WLOG let $a_1 \le c$, otherwise just consider a later segment of the sequence where this is true. Let N be the least index such tha $S_N \ge c$, and $S_{N-1} < c$. Put the terms in S_{N-1} in the subsequence $\{a_{n_k}\}$. Later terms will be picked as follows: pick the largest available a_n such that the running sum of ther terms added to $\{a_{n_k}\}$ is less than c. We can always find such a term because $\lim a_n = 0$.

The $\lim_{k\to\infty} S_{n_k} = c$. Fix $\epsilon > 0$. We'll show that exists K such that $S_{n_K} \ge c - \epsilon$. So we've picked the terms in S_{N-1} to be in $\{a_{n_k}\}$. If $S_{N-1} \ge c - \epsilon$, then we're done. If $S_{N-1} < c - \epsilon$, let $c' = c - \epsilon - S_{N-1}$. Again, there's an index N' such that $n \ge N'$ means $a_n \le c'$. Consider the subsequence $\{a_{N'}, a_{N'+1}, a_{N'+2}, ...\}$. Since there are only finite many terms before $a_{N'}$, the subsequence above have sum that diverges to infinite and each term in the above subsequence have entries no larger than c'. So there exsits a least index N'' such that $a_{N'} + a_{N'+1} + ... + a_{N''} \ge c'$. Since we're picking the largest available a_n to be in $\{a_{n_k}\}$, each new term added to $\{a_{n_k}\}$ which add up to $c - \epsilon$ can only be greater than or equal to each of the $a_{N'} + a_{N'+1}, ..., a_{N''}$, so there must be a K such that $S_{n_K} \ge c - \epsilon$. So we're done. □

2) Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be bounded sequences, and define the sets

$$A := \{a_n\}, B := \{b_n\}, C := \{a_n + b_n\}$$

Prove or provide a counterexample each of the following statements.

a) If $a \in \mathbb{R}$ is a limit point for A and $b \in \mathbb{R}$ is a limit point for B, then $a + b$ is a limit point for C.

Proof. False. Consider $\{a_n\}_{n=1}^{\infty} = \{0, 2, 0, 2, 0, 2, ...\}$ and $\{b_n\}_{n=1}^{\infty} = \{3, 0, 3, 0, 3, 0, ...\}$. It's obvious that 2 is a limit point of A, 3 is a limit point of B. But 5 is not a limit point of C because $\{a_n + b_n\}_{n=1}^{\infty} = \{3, 2, 3, 2, 3, 2, ...\}$ with limit points 3 and 2. □

6) The parts of this problem are not connectec. a) Let $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ and a strictly increasing sequence $\{x_n\}_{n=1}^\infty \subset (0,1)$ be given. Assume that $\sum_{n=1}^\infty a_n$ is absolutely convergent, and define $\alpha : [0,1] \to \mathbb{R}$ by

$$\alpha(x) := \begin{cases} a_n & x = x_n \\ 0 & \text{otherwise} \end{cases}$$

Prove or disprove: α has bounded variation on $[0,1]$.

Proof. Yes. Fix any partition P. Let N be the largest index such that $x_N \in P$. We have

$$V(P,\alpha) = \sum_{i=1}^n |\alpha(p_i) - \alpha(p_{i-1})|$$

If $p_i, p_{i-1} \notin \{x_n\}$, then we have $|\alpha(p_i) - \alpha(p_{i-1})| = 0$. If only one of p_i, p_{i-1} is in $\{x_n\}$, WLOG say $p_i = x_n$. Then $|\alpha(p_i) - \alpha(p_{i-1})| = |a_n| \le 2|a_n|$. If both $p_i, p_{i-1} \in \{x_n\}$, say $p_i = x_k, p_{i-1} = x_j, j < k$. Note $|\alpha(p_i) - \alpha(p_{i-1})| = |a_k - a_j| \le 2\max(|a_k|, |a_j|) \le 2|a_k| + 2|a_j|$. So then we have

$$\sum_{i=1}^n |\alpha(p_i) - \alpha(p_{i-1})| \le \sum_{i=1}^N 4|a_i| \le 4\sum_{i=1}^\infty |a_i| < \infty$$

By our reasoning above, if $|\alpha(p_i) - \alpha(p_{i-1})|$ covers any x_n's, it's smaller than twice the absolute value of the x_n's it covers. Each a_i could be appear in the sum twice, because point x_i is used twice in the partition. Hence the factor of 4 in the sum. It might not cover every x_n upto the largest index covered N, hence the first inequality. The last inequality is because $\{a_n\}$ converges absolutely. This is for an arbitrary P, so α is of bounded variation. □

b) Suppose that $f : [0,1] \to \mathbb{R}$ is Riemann-Stieltjes integrable with respect to a non-decreasing function $\beta : [0,1] \to [0,\infty)$. Prove that f is Riemann-Stieltjes integrable with resepct to the function β^2.

Proof. Since β is monotone increasing and x is continuous, by Theorem 6.9, $\beta \in \mathcal{R}(x)$, which implies that β is bounded, i.e. $|\beta| = \beta < B$ for some real B (the last implication is also used in the first sentence of the proof of theorem 6.20, so proof omitted here).

Alternate argument for β being bounded: Since β monotone on [0,1], β has bounded variation, which implies it's bounded. By theorem 6.6, to show $f \in \beta$, we'll show for all $\epsilon > 0$, there is a partition P such that $U(P,f,\beta^2) - L(P,f,\beta^2) < \epsilon$.

$$\begin{aligned} U(P,f,\beta^2) - L(P,f,\beta^2) &= \sum(M_i - m_i)(\beta_i^2 - \beta_{i-1}^2) \\ &= \sum(M_i - m_i)(\beta_i - \beta_{i-1})(\beta_i + \beta_{i-1}) \\ &= \sum(M_i - m_i)(\beta_i - \beta_{i-1})\beta_i + \sum(M_i - m_i)(\beta_i - \beta_{i-1})\beta_{i+1} \\ &\le 2B\sum(M_i - m_i)(\beta_i - \beta_{i-1}) \end{aligned}$$

Since $f \in \mathcal{R}(\beta)$, there is a P such that $\sum(M_i - m_i)(\beta_i - \beta_{i-1}) < \frac{\epsilon}{2B}$. This P works.

□

2019 May

2) Let $\{a_n\}_{n+1}^{\infty} \subset (0,\infty)$ be given, and assume that $\sum_{n=1}^{\infty} a_n$ converges.

a) Show that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges and $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ diverges.

Proof.

$$|\frac{a_n}{1+a_n}| = \frac{1}{1+a_n} a_n \leq a_n$$

All because $a_n > 0$. Then by comparison test, the desired series converges.

Since $\sum_{n=1}^{\infty} a_n$ converges, by theorem 3.23, $\lim a_n = 0$. So for $\epsilon = 1$, there exist N such that all $n > N$ have $a_n < 1$, so $1 + a_n < 2$, so $\frac{1}{1+a_n} > \frac{1}{2}$. And $\sum \frac{1}{2}$ diverges. Then by comparison test, the desired series diverges. □

b) Suppose that $\{b_n\}_{n=1}^{\infty} \subset \mathbb{R}$ satisfies $|b_{n+1} - b_n| \leq a_n$, for every $n \in \mathbb{N}$. Prove $\{b_n\}_{n=1}^{\infty}$ is convergent.

Proof. If we can show that $\{b_n\}_{n=1}^{\infty}$ is cauchy, then we've shown it's convergent. To show cauchy: $\forall \epsilon > 0, \exists N$ such that for all $m \geq n \geq N$ implies $|b_n - b_m| < \epsilon$. Fix $\epsilon > 0$.

Since $\sum_{n=1}^{\infty} a_n$ converges, we know that exists N_a such that for all $m \geq n \geq N_a$ implies

$$|a_n + a_{n+1} + ... + a_m| = a_n + a_{n+1} + ... + a_m < \epsilon$$

Let $N = N_a$, then we have

$$\begin{aligned}
|b_n - b_m| &= |b_n - b_{n+1} + b_{n+1} - b_{n+2} + b_{n+2} - b_{n+3} + b_{n+3} - ... - b_{m-1} + b_{m-1} - b_m| \\
&= |b_n - b_{n+1}| + |b_{n+1} - b_{n+2}| + |b_{n+2} - b_{n+3}| + |b_{n+3} - ... - b_{m-1}| + |b_{m-1} - b_m| \\
&\leq a_n + a_{n+1} + ... + a_{m-1} \\
&< \epsilon
\end{aligned}$$

as desired.

□

2018 January

3) Let $\{a_n'\}$ be any rearrangement of an infinite sequence $\{a_n\}$. Assume $\sum a_n$ converges absolutely. Prove $\sum a_n' = \sum a_n$.

Proof. ✓

We'll show $\sum a_n' - \sum a_n = 0$, which is equivalent to showing the sequence of partial sums $\{s_n - s_n'\} \to 0$. Fix $\epsilon > 0$. Since $\sum a_n$ converges absolutely, exsits N such that $m, n > N$ implies $\sum_m^n |a_i| < \epsilon$. Fixing $n = N+1$ and letting m go to infinity, we would get $\sum_{N+1} |a_n| < \epsilon$. Let N' be large enough such that $\{a_n'\}_{n \le N'} \supset \{a_n\}_{n \le N}$. Let $M = \max(N', N)$. So $s_M - s_M'$ only has terms with indices greater than N, so $|s_M - s_M'| \le |\sum_{N+1} a_n| \le \sum_{N+1} |a_n| < \epsilon$, as desired. □

5) Let $f : [a,b] \to \mathbb{R}$. Suppose $f \in BV[a,b]$. Prove f is the difference of two increasing functions.

Proof. Let $U(x) = V_a^x(f), L(x) = V_a^x(f) - f(x)$. Obviously, $U(x) - L(x) = f(x)$. Now just prove they're increasing. Let $x_1 \le x_2$. $U(x)$ is the "accumulation" of vertical distance traversed by f, so it's only increasing. In other words, $U(x_2) - U(x_1) = V_{x_1}^{x_2}(f) < \infty$ since $f \in BV[x_1, x_2] \subset BV[a,b]$.

$V_a^{x_2}(f) - V_a^{x_1}(f) = V_{x_1}^{x_2}(f) \ge |f(x_2) - f(x_1)| \ge f(x_2) - f(x_1)$. Rearranging we have $L(x_2) = V_a^{x_2}(f) - f(x_2) \ge V_a^{x_1} - f(x_1) = L(x_1)$, as desired. □

2017 January

2) Define $f : [0,1] \to [-1,-1]$ by

$$f(x) = \begin{cases} x\sin(1/x) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

a) Determine, with justification, whether f is of bounded variation on the interval $[0,1]$.

Proof. Note $\sin(x)$ is 1 at $\frac{(2n+1)\pi}{2}$ and 0 at $\frac{(2n)\pi}{2}$. So let partition $P_n = \{0, \frac{2}{n\pi}, \frac{2}{(n-1)\pi}, ..., \frac{2}{\pi}, 1\}$, then we'd have $\sin(1/x_i)$ oscillating between 0,1. WLOG say $\sin(\frac{2}{n\pi}) = 1$. Then

$$\begin{aligned}\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| &= |\frac{2}{n\pi}\sin(\frac{n\pi}{2}) - 0| + \sum_{i=2}^{n-1} |\frac{2}{i\pi}\sin(\frac{i\pi}{2}) - \frac{2}{(i-1)\pi}\sin(\frac{(i-1)\pi}{2})| + |\frac{2}{\pi}\sin(\frac{\pi}{2}) - \sin(1)| \\ &\geq |\frac{2}{(n-2)\pi}| + |\frac{2}{(n-2)\pi}| + |\frac{2}{(n-4)\pi}| + |\frac{2}{(n-4)\pi}| + ... + |\frac{2}{\pi}| \\ &= \frac{2}{\pi}(\frac{2}{n-2} + \frac{2}{n-4} + ... + 1) \\ &\geq \frac{2}{\pi}(\frac{1}{n-1} + \frac{1}{n-2} + \frac{1}{n-3} + \frac{1}{n-4} + ... + 1)\end{aligned}$$

Note as we let n go to infinity, the above sum diverges. So the total variation of f is not finite. □

b) Determine, with justification, whether f is continuous on the interval $[0,1]$.

Proof. Note $1/x$ is continuous on $(0,1]$, as well as $\sin(x)$, so by theorem 4.7, $f(x)$ is continuous on $(0,1]$. Now to determine if it's continuous at 0, lets see if $\lim_{x\to 0} f(x) = f(0) = 0$. If it is, then by definition of continuity, it's continuous at 0. Fix $\epsilon > 0$. Let $\delta = \epsilon$. Then if $|x| < \delta$, we have

$$|x\sin(1/x)| \leq |x| < \delta = \epsilon$$

as desired. So f is continuous on $[0,1]$

□

2015 Jan 21

1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Fix $c \in \mathbb{R}$, and suppose that f has the following property: there is an L such that for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|\frac{f(r) - f(c)}{r - c} - L| < \epsilon$$

whenever $r \in \mathbb{Q}$ and $0 < |r - c| < \delta$. Prove that f is differentiable at c and that $f'(c) = L$.

Proof. By definition 5.1, we'll show that for $t \in \mathbb{R}, t \neq c, \lim_{t\to c} \frac{f(t)-f(c)}{t-c} = L$. Unfolding the definition of the limit, we wish to show that $\forall \epsilon > 0, \exists \delta > 0$ such that $|\frac{f(t)-f(c)}{t-c} - L| < \epsilon$.

Fix $\epsilon > 0$. By assumption of the question, there is a $\delta_1 > 0$ such that

$$|\frac{f(r) - f(c)}{r - c} - L| < \epsilon/2$$

whenever $r \in \mathbb{Q}$ and $0 < |r - c| < \delta_1$. If $t \in \mathbb{Q}$, then let $|t - c| < \delta_1$, we'd have $|\frac{f(t)-f(c)}{t-c} - L| < \epsilon/2 < \epsilon$, as desired.

Now suppose that t is not rational. We want to show that there is a rational r close to t such that $|\frac{f(t)-f(c)}{t-c} - \frac{f(r)-f(c)}{r-c}| < \epsilon/2$. Since $f(x)$ is continuous on $\mathbb{R}$, we know that $f(x) - f(c)$ is also continuous on $\mathbb{R}$. $\frac{1}{x-c}$ is continuous on $\mathbb{R}\backslash\{c\}$, so we know $\frac{f(x)-f(c)}{x-c}$ is continuous on $\mathbb{R}\backslash\{c\}$ as well. So we have $\lim_{y\to x} \frac{f(y)-f(c)}{y-c} = \frac{f(x)-f(c)}{x-c}$. By definition of the limit, we know that for $\epsilon/2$, exists $\delta_2 > 0$ such that $|y - x| < \delta_2$ implies $|\frac{f(y)-f(c)}{y-c} - \frac{f(x)-f(c)}{x-c}| < \epsilon/2$.

Note $t \neq c$ by set up. Since the rational numbers are dense in the reals, t is a limit point of the rational numbers. So for some $\delta_2 \in (0, |t - c|)$ there is an $r \in \mathbb{Q}$ in the neighborhood $N_{\delta_2}(t)$, i.e. $|r - t| < \delta_2$. And note that $r \neq c$ also because $\delta_2 < |t - c|$. So we can apply our result in the above paragraph, we have

$$|\frac{f(r) - f(c)}{r - c} - \frac{f(t) - f(c)}{t - c}| < \epsilon/2$$

Now let $\delta = \min\{\delta_1, \delta_2, \delta_1 - \delta_2\}$. And we insist $|t - c| < \delta \leq \delta_1 - \delta_2$ Recall we had $|r - t| < \delta_2$ so $|r - c| \leq |r - t| + |t - c| < \delta_1 - \delta_2 + \delta_2 = \delta_1$. So by assumption we also have

$$|\frac{f(r) - f(c)}{r - c} - L| < \epsilon/2$$

Combining the above two inequalities, we have that if $|t - c| < \delta$, then

$$|\frac{f(t) - f(c)}{t - c} - L| \leq |\frac{f(t) - f(c)}{t - c} - \frac{f(r) - f(c)}{r - c}| + |\frac{f(r) - f(c)}{r - c} - L| < \epsilon/2 + \epsilon/2 = \epsilon$$

So $\delta = \min\{\delta_1, \delta_2, \delta_1 - \delta_2\}$ works for both when f is rational or irrational.

□

2012 Jan 19

Question 1 ✓

a) Let $(a_n)_{n \in N}$ and $(b_n)_{n \in N}$ be bounded sequences of positive real numbers. Suppose that $\sum b_n$ is convergent. Show that $\sum a_n b_n$ is also convergent.

Proof. Since they're bounded, $a_n < M$ for all n for some M. And

$$\sum a_n b_n < \sum M b_n$$

The right hand side is convergent. Since both series have positive entries, $|a_n b_n| = a_n b_n < M b_n$. By comparison test, the LHS converges. □

b) Let $y \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given. Suppose for every sequence (x_n) we have $\inf |f(x_n) - f(y)| \leq \inf |x_n - y|$. Prove that f is continuous at y.

Proof. Recall theorem 4.2: $\lim_{x \to y} f(x) = f(y)$ if and only if $f(x_n) = f(y)$ for every sequence $\{x_n\}$ with $x_n \neq x$ but $x_n = y$.

Fix some sequence $x_n \to y$. So we know that $|x_n - y| = 0$. Note $|x_n - y| = \inf |x_n - y|$, so we have

$$0 \geq \inf |f(x_n) - f(y)| \geq 0$$

The last inequality is because every term is positive in the absolute value sign. So we know $\inf |f(x_n) - f(y)| = 0$. By the definition of lim inf, this means that out of all subsequential limits, the lowest one could only be as low as 0. Now we show that there're no subsequencial limits strictly greater than 0. AFSCO, $\{x_{n_k}\}$ is a subseq with $\lim_{k \to \infty} |f(x_{n_k}) - f(y)| = \epsilon > 0$. Note since $\{x_{n_k}\}$ is a subsequence, it has the same limit as $\{x_n\}$. So $\lim_{k \to \infty} |x_{n_k} - y| = 0$. Again, recall if the limit exsits, then the limit is equal to the lim inf. Now we have

$$\lim_{k \to \infty} \inf |x_{n_k} - y| = 0 \geq \lim_{k \to \infty} \inf |f(x_{n_k}) - f(y)| = \epsilon > 0$$

We have shwon $0 > 0$, which is absurd. So there can't be any other subsequence with $\lim_{k \to \infty} |f(x_{n_k}) - f(y)| > 0$, so the $|f(x_n) - f(y)| = 0$, implying $f(x_n) = f(y)$, as desired. So f is continuous. □