Question

Let G be a connected regular bipartite graph with bipartition X, Y. Prove that for all $x \in X$ and $y \in Y$ the graph $G' = G \setminus \{x, y\}$ has a perfect matching.

Non-constructive solution: We'll show G' satisfies Hall's condition. Suppose $S \subseteq X$ has $N_{G'}(S) = T$ with $|T| < |S|$. We know that in G there are $d|S|$ edges leaving S, and they must all go to $T \cup \{y\}$, a set of size at most $|S|$. Since at most $d|S|$ edges can arrive there, it must be that $N_G(S) = T \cup \{y\}$ and $N_G(T \cup \{y\}) = S$. But then $S \cup T \cup \{y\}$ is a component of G not containing x, a contradiction.

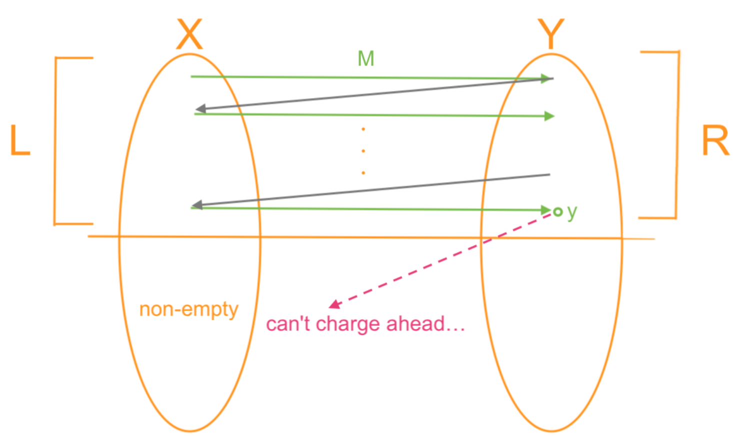
Constructive solution: Let G be k-regular. By a corollary of Hall's theorem, G being regular biparite means G has a perfect matching M and $|X| = |Y|$ by counting edges.

If $xy \in M$, then we're done. Just remove xy from M and the remaining matching is a perfect matching on the rest of the vertices.

Now suppose that $xs \in M, ty \in M$ where $s \neq y, t \neq x$. Since G is connected, there exists a path from s to t. In particular, I can choose this st-path to be M-alternating. In fact, I'm going to show that we can visit every vertex in G by going on one M-alternating walk, which implies the previous sentence.

Claim: Start from X. Going from X to Y, put the unique M-edge in our walk. Going from Y to X, pick any non-M edge to travel back to X, we can visit all the vertices in this way.

Pf of claim:



Let $L \subseteq X$ be the vertices visited in X, let $R \subseteq Y$ be vertices visited in Y. Since we get no choice going from X to Y, the burden of expanding the frontier of our territory falls on the choices the Y vertices make. Of course we want to keep seeing new vertices so that we're guaranteed to visit all the vertices during our walk.

All is well unless if we stand on $y \in Y$ and we've got more vertices to see (i.e. $X \backslash L \neq \varnothing$), but all our edges extend into the old vertices. We can phone a friend in set R. Since we've walked down in zig-zag, we can walk back up our M-alternating path to find any vertex in R. If any one vertex in R can get into the new territory, we're good, we can keep going. A terrible scenario would be that none of R can get into unchartered land, that means all $k|R|$ edges go into L. And since for each vertex in R, it must come from a vertex in L via an M edge, and we'd said that we've landed in Y where this whole ordeal started, so $|L| = |R|$. So the average number of edges from R that each vertex in L is incident to is $\frac{k|R|}{|L|} = k$.

Since each vertex can only be incident to at most k edges because G is k-regular, we know that each vertex in L is incident to exactly k edges, all coming from/going into R. So the subgraph $L \cup R$ is not conneced to the rest of the graph in G, so G is disconnected, which is a contradiction. So our imaged dilemma of being stuck in Y while there're more vertices to see is non-existent. So we can get to all vertices in X. And by the highway paved by M-edges, if we see vertices in X, we can travel through the M-edges to see their matched vertex in Y. Since M is a perfect matching, we also see everyone in Y. So our M-alternating walk sees everyone. (end pf of claim)

In particular, we have an M-alternating walk from s to t, from which we can extract an M-alternating path, denoted $P = sp_1...p_n t$.

Note $xPy = xsp_1...p_n ty$ is an M-alternating path starting and ending with M-edges. Now we discard the M-edges in xPy, and add the non-M edges to M. Now x and y are not matched because edges xs and ty left the matching, but everyone else in $\{s, p_1, ..., p_n, t\}$ are still matched.

For other vertices not in the xPy path, their matching status hasn't changed. We can now removed the unmatched x and y, and the rest of the vertices are perfectly matched, as dicussed above.