# June 2023

**Question 1** Recall that if $\pi = \pi_1\pi_2...\pi_n$ is a permutation of $[n]$ then we say that the pair of indices $(i, j)$ is an inversion of $\pi$ if $i < j$ but $\pi_i > \pi_j$. For $n \geq 1$ let $I_n$ be the total number of inversions in all permutations of $[n]$. For instance, $I_1 = 0, I_2 = 1, I_3 = 9$.

a) Prove that for all positive integers $n$ we have

$$I_{n+1} = (n+1)I_n + n!\binom{n+1}{2}$$

*Proof.* We'll partition the counting in to the position of the element $n+1$. Say if $n+1$ is in position $i$, then for each of the $n!$ permutation $\pi$ of the rest of the $n$ elements, the number of inversion of that specific permutation is $inv(\pi) + i$. Now let's sum over all $n+1$ positions that element $n+1$ could be. Let's denote $\pi_i$ to be permutation of $[n+1]$ with $n+1$ in position $i$. And note since the position of $n+1$ is fixed, $\pi_i$ is essentially a permutation of $[n]$

$$\begin{aligned}
\sum_{i=1}^{n+1} \sum_{\pi_i \in \Sigma_n} (inv(\pi_i) + (n+1) - i) &= \sum_{i=1}^{n+1} \sum_{\pi_i \in \Sigma_n} inv(\pi_i) + \sum_{i=1}^{n+1} \sum_{\pi_i \in \Sigma_n} (n+1) - i) \\
&= \sum_{i=1}^{n+1} I_n + \sum_{i=1}^{n+1} n!(n+1-i) \\
&= (n+1)I_n + n! \sum_{k=0}^{n} k \\
&= (n+1)I_n + n! \frac{n(n+1)}{2} \\
&= (n+1)I_n + n!\binom{n+1}{2}
\end{aligned}$$

□

b) Use the aboverecurrence to deduce that for all $n \geq 1$

$$I_n = \frac{n!}{2}\binom{n}{2}$$

*Proof.* Look for the pattern. Iteratively substitute $I_n$ into $I_{n+1}$ and simplify to get

$$\begin{aligned}
I_{n+1} &= (n+1)nI_{n-1} + \frac{(n+1)!}{n}\binom{n}{2} + \frac{(n+1)!}{n+1}\binom{n+1}{2} \\
&= (n+1)n(n-1)I_{n-2} + \frac{(n+1)!}{n-1}\binom{n-1}{2} + \frac{(n+1)!}{n}\binom{n}{2} + \frac{(n+1)!}{n+1}\binom{n+1}{2} \\
&\dots \\
&= \sum_{k=1}^{n+1} \frac{(n+1)!}{k}\binom{k}{2}
\end{aligned}$$

The above pattern can be proven with induction which is obvious, so omitted. So we obtain the equation $I_n = \sum_{k=1}^{n} \frac{n!}{k}\binom{k}{2}$.

$$\begin{aligned}\sum_{k=1}^{n} \frac{n!}{k}\binom{k}{2} &= n!\sum_{k=1}^{n} \frac{1}{k}\frac{k(k-1)}{2} \\ &= n! 0.5 \sum_{i=0}^{n-1} i \\ &= n! 0.5 \frac{n(n-1)}{2} \\ &= \frac{n!}{2}\binom{n}{2}\end{aligned}$$

as desired. □

**Question 3** a) State and provethe Orbit Counting Lemma (sometimes called Burnside's Lemma), relating the number of orbits in a group action $G \rightarrowtail X$ to the number of fixed points of the various group elements $g \in G$

*Proof.* Claim: Let $G \rightarrowtail X$ with $G, X$ finite. For all $x$, we have $Orb(x) \cdot Stab(x) = |G|$.

Proof of claim: Let $Orb(x) = \{x_1, ..., x_k\}$. Let $\{g_1, ..., g_k\}$ be some corresponding elements such that $g_i x = x_i$. Define function

$$m : (g_i, h) \mapsto g_i h$$

where $g_i \in \{g_1, .., g_k\}, h \in Stab(x)$. Showing $m$ to be a bijection would show the desired equation.

To show $m$ is 1-1. Suppose $g_i h = g_j h'$. Then we know

$$\begin{aligned} g_i h x &= g_j h' x \\ g_i x &= g_j x \\ x_i &= x_j \end{aligned}$$

So $i = j$ and $g_i = g_j$. Also note $h = g_i^{-1}(g_i h) = g_j^{-1}(g_j h') = h'$ So we showed $(g_i, h) = (g_j, h')$.

To show $m$ is onto, fix $g \in G$. Suppose $gx = x_i = g_i x$. Let $h = g_i^{-1} g$. Then we have

$$g_i h x = g_i (g_i^{-1} g) x = gx$$

So $(g_i, g_i^{-1} g) \mapsto g$ as desired.
(end of proof of claim)

The Orbit Counting Lemma states that $\#Orb = \frac{1}{|G|}\sum_{g\in G} Fix(g)$. Let $O$ be the set of orbits.

$$\begin{aligned}
\#Orb &= \sum_{T\in O} 1 = \sum_{T\in O}\sum_{x\in T}\frac{1}{|T|} = \sum_{T\in O}\sum_{x\in T}\frac{1}{|Orb(x)|} \\
&= \sum_{x\in X}\frac{1}{|Orb(x)|} \\
&= \frac{1}{|G|}\sum_{x\in X}|Stab(x)| \\
&= \frac{1}{|G|}\sum_{x\in X}\sum_{g\in G, g\cdot x = x} 1 \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{x\in X, g\cdot x = x} 1 \\
&= \frac{1}{|G|}\sum_{g\in G} Fix(g)
\end{aligned}$$

as desired □

b) How many distinguishable 10 bead necklaces can be made using $k$ colors of beads?

*Proof.* We'll apply the orbit counting lemma:

$$\#Orb = \frac{1}{|G|}\sum_{g\in G} Fix(g)$$

where $G$ is the group of rotational symmetries, i.e. $\mathbb{Z}/10$. Let $G$ act on the number of colorings, i.e. $G \rightarrowtail K^{10}$. Given a coloring of the necklace, rotating it would yield the same/equivalent coloring, in other words, colorings that are in the same orbit are equivalent/the same. So we just count the number of orbits under this action. What we really need to do is count the number of colorings that are fixed/looks the same when you rotate it by $g$.

Say you rotate by 2 clicks at a time. Since rotating would yield the same coloring, this means the beads $\{1,3,5,7,9\}$ would have the same color, and so do beads $\{2,4,6,8,10\}$. These are cycles under the permutation given by the rotation by 2 clicks. Each cycle you can color it $k$ colors, so here we'd have $k^2$ colorings that are fixed under rotating by 2 clicks.

Generalizing, with any rotation $g$, we'd have $k^{\text{number of cycles in } \pi(g)}$ number of colorings that are invariant. I happen to know this fact that the number of cycles in rotating by $r$ clicks is the gcd(10,r). So with this fact, we obtain that

$$\begin{aligned}
\#Orb &= \frac{1}{|G|}\sum_{g\in G} Fix(g) \\
&= \frac{1}{10}(k^{gcd(0,10)} + k^{gcd(1,10)} + k^{gcd(2,10)} + k^{gcd(3,10)} + k^{gcd(4,10)} + k^{gcd(5,10)} \\
&\quad + k^{gcd(6,10)} + k^{gcd(7,10)} + k^{gcd(8,10)+k^{gcd(9,10)}} \\
&= \frac{1}{10}(k^{10} + k^1 + k^2 + k^1 + k^2 + k^5 + k^2 + k^1 + k^2 + k^1) \\
&= \frac{1}{10}(k^{10} + 4k + 4k^2 + k^5)
\end{aligned}$$

□

**Question 6** Recall that we call a family of subsets $F \subset P(n)$ a $k - family$ if no $k+1$ sets $A_1, A_2, ..., A_{k+1} \in F$ satisfy

$$A_1 \subsetneq A_2 \subsetneq ... \subsetneq A_{k+1}$$

Let $F$ be a k-family in $P(n)$.
a) Prove that if $\sigma$ is a circular permutation of $[n]$ then at most $kn$ sets from $F$ are intervals with respect to $\sigma$.

*Proof.* (improved proof) Note an interval has to be non-empty. So lets say the worst case we start with singleton intervals and try to stuff as many intervals into $F$ as we can. There are $n$ singleton intervals, and we can expand by appending to it the next element in the cycle, since it's a $k$ family, we can have sets at most $k$ big. So for each singleton interval, we can expand $k$ times and add to $F$ the sets we meet along the way. There are $n$ singleton intervals. So we can add $kn$ things into $F$ in the worst case. The better case is that the k-family $F$ does not contain some smaller intervals, say the smallest interval is like size 3, so then we can expand $k-3$ times. That's why it's $\leq kn$ sets.

□

*Proof.* (original proof) Let's count the total number of sets in $F$ that are intervals with respect to $\sigma$ and that can also be in a k-family. Define $M$ be the set of minimal elements in $F$ that are intervals of $\sigma$ and such that each element in $M$ is in a different chain, i.e. for $x, y \in M, x \notin y$.

We'll show that $|M| \leq n$. WLOG fix the min number in $\sigma$ to be in the 1st position. WLOG, for $x \in M$, we order the elements in $x$ in the order they appear in $\sigma$. We will try to "fit" the $x \in M$ into the $\sigma$ one by one following an increasing order of the earliest position in $\sigma$. WLOG say an $x \in M$ starts in position one, for the next set, since the $y \in M$ are mutually not a subset of the others, the ending position of $y$ will have to be after the ending position of $x$. Say the ending position of $x$ is the $l$-th position. If we want to pack as many as possible, we should only advance the ending position of the interval 1 at a time. We can do that AT MOST $n$ times before the ending position will loop around and pass the position $l$. This is a problem because we pack the intervals in the order of their starting position, so if an interval that comes later than the first interval has ending position past $l$ (the ending position of the first interval), this means that the first interval is a subset of that interval, which contradicts the construction of $M$.

Note for each $x \in M$, we can obtain a new set to add to $F$ by adding the next number in $\sigma$ that comes after $x$. Since $F$ is a $k$-family, we can do that at most $k$ times. So each $x \in M$ could bring at most $k$ things that we can add to $F$. Since there are at most $n$ things in $M$, we can add at most $kn$ things to $F$.

□

b) Prove that if $F$ contains $a_i$ sets of size $i$ then

$$\sum_{i=0}^{n} \frac{a_i}{\binom{n}{i}} \le k.$$

[Hint: you might want to first prove the case where $\varnothing, [n] \notin F$.]

*Proof.* (improved proof) We'll apply the LYM inequality. Recall the LYM inequality says that given an antichain $A$, we have $\sum_{i=0}^{n} \frac{a_i}{\binom{n}{i}} \le 1$. Let $M_1$ be the set of minimal elements in $F$ such that they're mutually not related, i.e. $M_1$ is an antichain. We can grow the $M_1$ upwards by adding the things in 1 level up that are also in $F$ and put those things in $M_2$. Note since $F$ is a k family, we can have at most $k$ of the $M_i$ that's all non-empty. Note each of the $M_i$ is an antichain (omit the detail here you can verify easily). So we have at most $k$ antichains. Apply LYM to the $k$ antichains and you will get desired answer.

□

*Proof.* (original proof) Define $C$ to be the set of maximal chains in $P(n)$. Note the size of $C$ is $n!$. Let set $S = \{(f, c) | f \in F, c \in C, f \in c\}$. Note we have

$$|S| = \sum_{c} \sum_{f \in F, f \in c} 1 \le \sum_{c} k = kn!$$

because $F$ is a k-family, so for any given chain, we can have at most $k$ things of $F$ in it. At the same time, we have

$$\begin{aligned} |S| &= \sum_{c} \sum_{f \in F, f \in c} 1 \\ &= \sum_{f \in F} \sum_{c : f \in c} 1 \\ &= \sum_{f \in F} |f|!(n - |f|)! \\ &= \sum_{i \ge 0} a_i i!(n-i)! \end{aligned}$$

The second equality is a simple interchange of sums. The thrid equality is to count how many maximal chains a given set can be in. We can permute the elements in the set to decide the path of the chain going towards the emptyset, and we can permute the elements not in the set to decide the path of the chain going towards the set $[n]$. The last equality is couting by the size of $f$.

Now combining the two expressions obtained above and dividing by $n!$, we get that $\sum_{i\geq 0} a_i/\binom{n}{i} \leq k$ as desired □

**Question 7**
a) State and prove Hall's theorem concerning the existence of matchings saturating $X$ in a bipartite graph with bipartition $X, Y$.

*Proof.* We'll use induction on $|X|$. It would be nice if we can just take off a vertex in $X$ and not affect the Hall's condition in the rest of the graph. That happens when every subset of $X$ has a neighborhood strictly larger than it. So we can take off a vertex in $X$ and one of its neighbors and Hall's condition still holds in the new graph.

The problematic case is when there is a set $S \subset X$ such that $|S| = N(S)$. Obviously the graph restricted to $(S, N(S))$ satisfies the Hall's condition so there's a perfect matching on the subgraph $(S, N(S))$. Now we just need to show in $(S, N(S))^c$, Hall's condition is also satisfied. Fix $T \subset X \backslash S$, if $N(T)$ doesn't intersect the $N(S)$, then its fine. Suppose they do intersect. The fear is that maybe some of $T$'s neighbor leaks into $N(S)$ and there is not enough neighbors left in $(S, N(S))^c$. Denote $N_1(T) = N(T) \cap N(S)$ and $N_2(T) = N(T) \cap N(S)^c$. Note $N(S \cup T) = N(S) \cup N_2(T)$. If our fear were to come true, we'd have $N_2(T) < |T|$, so we'd have $|S \cup T| < N(S) + N_2(T)$, this is a contradiction to our assumption. □

**Question 8** Let $G$ be a 2-connected graph. Prove that for all $v \in V(G)$ there exists $u \in V(G)$ such that $u$ is adjacent to $v$ and also $G \backslash \{u, v\}$ is connected.

*Proof.* Fix $v \in V(G)$. Since $G$ is 2-connected, removing $v$ is still connected. If a $u \in N(v)$ disconnects the new graph upon removal, it is a cut vertex in the new graph. AFSOC all of $u \in N(v)$ disconnects the new graph upon removal, this means they're all cut vertices in the new graph. I claim that the $u_1$, the cut vertex at either end of the block-cut vertex graph, is also a cut vertex in the old graph. Say this $u_1$ connects end block $B$ to the rest of the graph denoted $B'$. If $u_1$ is not a cut vertex in the old graph, it must mean that $v$ has tentacles tethering the 2 parts together, so $v$ is attached to some internal vertex of the 2 parts, i.e. $v$ is nbors with an interval vertex in block $B$. But we literally just said all of $v$'s neighbors are cut vertices, contradiction.

□

**Question 9** Let $G$ be a graph that does not contain two disjoint odd cylcles. Prove that $\chi(G) \leq 5$.

*Proof.* If $G$ has no odd cycles, then it's bipartite, so it's 2-colorable.

If $G$ has 1 odd cycle, let the smallest odd cycle be $C$, the odd cycle is 3-colorable. And $G \backslash C$ is bipartite, so it's 2 colorable. So we can color the whole graph with 5 colors. If it doesn't have odd cycle, it's bipartite and 2-colorable.

$\square$

## June 2022

**Question 1** a) Prove the following identity for any positive integers $m, n$ and any nonnegative integer $r$:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Note: Here $\binom{s}{t} = 0$ if $s < t$ or $t < 0$ by convention.

*Proof.* The two sides of the equation count picking $r$ distinguished m&m's from $m+n$ distinguised m&m's, where $m$ of them are magenta and $n$ of them are navy. Obviously the left counts this. Now we prove the right also counts this. It separates into cases where there are $k$ m&m's that are magenta, each term in the sum is a case. From the navy m&m's, we pick the rest of the $r-k$ m&m's. □

b) For an indeterminate $q$, we define the "$q-integer$" $[n]_q$ as $[n]_q := 1 + q + ... + q^{n-1}$ if $n$ is a positive integer and $[0]_q := 0$. The "$q-factorial$" is defined as the product of consecutive q-integers as $[n]_q! := [1]_q[2]_q...[n]_q$ if $n$ is a positive integer, and $[0]_q! := 1$. For a positive integer $s$ and an integer $t$, the "$q-binomial\ coefficient$" is defined by

$$\binom{s}{t}_q := \frac{[s]_q!}{[t]_q![s-t]_q!}$$

if $s \geq t \geq 0$, and $\binom{s}{t}_q := 0$ if $s < t$ or $t < 0$.

Prove or disprove the following $q$-analog of the identity in part a) (for positive integers $m, n$ and nonnegative integer $r$):

$$\binom{m+n}{r}_q = \sum_{k=0}^{r} \binom{m}{k}_q \binom{n}{r-k}_q q^{k(n-r+k)}$$

**Question 9**

Let $G$ be a connected graph with chromatic number at least $k+1$, for some positive integer $k$. Prove that one can remove $\frac{k(k-1)}{2}$ edges from $G$ without disconnecting it.

*Proof.* Note between any 2 color class there is an edge. So we have a $K_{k+1}$ graph of color classes. To remain connected you can remove all the edges and add back in $k$ edges each connecting the color classes $i, i+1$. So you're removing $\frac{(k+1)k}{2} - k = \frac{k(k-1)}{2}$ number of edges. □

## May 2021

**Question 5** a) State (but you need not prove) Erdos-Ko-Rado's theorem.

*Proof.* The Erdos-Ko-Rado theorem gives a bound on the intersecting family of sets of size $k$. It says for $1 \leq k \leq \frac{n}{2}$ and $I \subset \binom{[n]}{k}$ intersecting, the size of $I$ is at most $\binom{n-1}{k-1}$. □

b) Construct a family of $r-$subsets of $[n]$ achieving the Erdos-Ko-Rado bound on cardinality.

*Proof.* Put $n$ in each of the $r-$subsets, from $[n-1]$, pick the remaining $r-1$ elements. There are $\binom{n-1}{r-1}$ elements in the family, and they are intersecting because $n$ is in every set. □

**Question 9** a) Let $T_1$ and $T_2$ be two spanning trees of a connected graph $G$. Prove that $T_1$ can be transformed into $T_2$ through a sequence of intermediate trees, each arising from the previous one by removing an edge and adding another.

*Proof.* Recall in class we proved for all $e \in T_1 \backslash T_2$, exists $e' \in T_2 \backslash T_1$ such that $T_1 - e + e'$ and $T_2 - e' + e$ are spanning trees. Let $D = T_1 \Delta T_2$ be the synmmetric difference of the edges in the 2 trees. Note $|D|$ is fintie. In $T_1$, iteraively replace an edge in $D \cap T_1$ with some edge in $T_2 \backslash T_1$. This replacement strictly decreases the size of $D$. Since $|D|$ finite, the process terminates and we end up with $D = \varnothing$, which implies $T_1 = T_2$.

□

## June 2020

**Question 2** A circle is divided into $p$ equal arcs, where $p$ is a prime number. Each arc is colored by one of the $n$ different colors. How many nonequivalent colorings are there? Two colorings are equivalent if one of them can be obtained from the other one by a rotation of the circle about its center.

*Proof.* We'll use the orbit counting lemma that states $\#Orb = \frac{1}{|G|}\sum_{g\in G} Fix(g)$. As explained above (2023 question 3) the orbits are the equivalence classes of colorings. Here the identity fixes every coloring so there are $n^p$ colorings. Rotating by any other amount would just yield 1 cycle because $p$ is prime (simple fact from undergraduate proof class). And each cycle must be colored the same color. So for all the other $p-1$ rotations, each would yield $n$ different colorings. So at the end we have the number of distinct colorings is

$$\frac{1}{p}(n^p + (p-1)n)$$

□

**Question 4**
a) State (without proving) Sperner's Lemma concerning the maximum size of an antichain in $p(n)$.

*Proof.* Sperner's lemma is a corollary of the LYM inequality. The LYM inequality states that the probability of landing in an antichain is spread across the levels, $\sum_{i=0}^{n} \frac{a_i}{\binom{n}{i}} \leq 1$. Since $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the biggest the denominator can get, $\frac{a_i}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \frac{a_i}{\binom{n}{i}}$. So we have $\frac{|A|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \sum_{i=0}^{n} \frac{a_i}{\binom{n}{i}} \leq 1$, which bounds the size of the antichain by $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. □

b) Let $a_1, ..., a_n$ be real numbers of absolute value at least one. For any open unit interval $I$, prove that there are atmost $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ vectors $\epsilon = (\epsilon_1, ..., \epsilon_n) \in \{-1, 1\}^n$ (i.e. $\epsilon$ is a vector whose entries are 1 or -1) such that $\sum_{i=1}^{n} \epsilon_i a_i \in I$.

*Proof.* For $\sum_{i=1}^{n} \epsilon_i a_i$, we attach to it the set $\{i | a_i \epsilon_i > 0\} \subset [n]$. Note this set contains the indices where $a_i \epsilon_i$ is positive, and actually greater than or equal to 1 because the $a_i$ have absolute value at least 1. Say $x, y \in P(n), x \subsetneq y$, we know that the corresponding sums $S_x, S_y$ would be $S_x = \sum_{i\in x} |a_i| - \sum_{j\notin x} |a_j| = \sum_{i\in x} |a_i| - \sum_{j\in y\setminus x} |a_j| - \sum_{k\notin y} |a_k|$ and $S_y = \sum_{i\in x} |a_i| + \sum_{j\in y\setminus x} |a_j| - \sum_{k\notin y} |a_k|$. You see that

$$|S_y - S_x| = |\sum_{i\in x} |a_i| + \sum_{j\in y\setminus x} |a_j| - \sum_{k\notin y} |a_k| - \sum_{i\in x} |a_i| + \sum_{j\in y\setminus x} |a_j| + \sum_{k\notin y} |a_k|| = 2 \sum_{j\in y\setminus x} |a_j| \geq 1$$

The last equality is because $x \neq y$. Note $|S_y - S_x| = |S_x - S_y|$, so this means that if sets $x, y$ are comparable, then the difference of their sum will be at least 1. The converse would imply that if the difference of the sum is $< 1$, then the two sets $x, y$ will not be comparable, in other words, $x, y$ are in an antichain. So given an open unit interval $I$, any vectors in it would be in an antichain with each other. By Sperner's lemma, there can be at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ of them.

□

**Question 7** The parts of this question are unrelated.

a) Let $G$ be a graph containing $k$ edge-disjoint spanning trees, and let $e_1, ..., e_k$ be distinct edges in $G$. Prove that $G$ has edge-disjoint spanning trees $T_1, ..., T_k$ such that $e_i \in T_i$ for $1 \leq i \leq k$.

*Proof.* We'll show the following algorithm will give us what we want. At each iteration $i$, label weight of the edges $e_i = 0, e_k = \infty$ for $k \neq i$ and the rest of the edges just label 1. Apply Kruskal's algorithm to obtain $T_i$. For the next iteration, delete the edges of $T_i$ from the graph and repeat. Note since $e_i$ has the minimum weight, it will be in the spanning tree $T_i$. It remains to show that we can in fact successfully run this algorithm for $k$ iterations.

The only criterion for the the algorithm to run successfully is that there is a spanning tree $T_i$ containing edge $e_i$ in the graph. What could go wrong is that there's no spanning tree or edge $e_i$ got deleted in earlier iterations. In the first case, it implies that the graph is not connected anymore, which means there is a set of edges of size $\leq i - 1 < k$ that is like a "cut edge" in the graph. This would make it impossible to have $k$ edge-disjoint spanning trees, so a contradiction.

Now let's suppose that edge $e_i$ got deleted in an earlier iteration $j < i \leq k$. In iteration $j$, edge $e_i$ got selected because it was the min weight edge that doesn't create a cycle when added to the tree. Note $e_i$ had weight infinity in iteration $j$, this means that all other (non-infinity) edges that are not one of the $e_l$'s must create a cycle when added. This implies that $e_i$ connect 2 distinct components of the current tree. Say $e_i = uv$, if the edge did not connect 2 distinct components, i.e. $e_i$ is within 1 component of the tree, then we know that $u \sim v$ in the currenct tree, so added $e_i$ would create a cycle, makes no sense to add it then. Now, the algorithm chose edge $e_i$ possibly due to tie breaking with other infinity-edges. So there might be other infinity edge that connect the 2 components that edge $e_i$ is connecting. Note $e_j$ ($j$ is our current iteration and $e_j$ has already been added as the first edge) cannot be connecting the 2 components because if it did then there would be a path from $u$ to $v$, which then again means it would make no sense to add $e_i$ now. There can also be no non-infinity edges connecting the 2 components, because the kruskal algorithm would have chosen it instead and we would never even consider edge $e_i$. So there are at most $k-1$ edges that connect the 2 components. Same argument as the previous paragraph, this implies that there can't be $k$ edge-disjoint spanning trees. □

**Question 9** Let $n \geq 6$ be even, and let $F = \{e_1, ..., e_{n-1}\}$ be a collection of $n-1$ different edges of $K_n$ that are not all incident with one same vertex. Show that $K_n - F$ has a perfect matching.

*Proof.* We induct on $n \geq 6$.

BC: The base case is small enough to do by inspection.

IH: Suppose for even $n \geq 6$, delete $n-1$ edges that are not all incident with one same vertex, then the remaining graph has a perfect matching.

IS: Let there be $n+2$ vertices with $n$ even. Delete $n+1$ edges. Consider the induced graph $H$ of the deleted edges, i.e. the deleted edges are edges in the induced graph. If $H$ contain a tree with $\geq 3$ vertices, and we know any tree would have 2 leaves. Delete the 2 leaves from the original graph. Now we end up with $n$ vertices. And since the tree has size $\geq 3$, we know the 2 leaves are not connected in $H$, so we also removes 2 deleted edges, so now there are only $n-1$ deleted edges. Now we can apply the IH to the remaining graph to obtain a perfect matching. Also note that since the 2 leaves we removed are not connected in $H$, we know it's connected in the original graph and just add the edge between the 2 vertices into the matching. Now we have a perfect matchiing for all the vertices.

Now suppose $H$ does not contain any tree. This would imply that $H$ does not invole all the $n+2$ vertices. Because if it did involve all the $n+2$ vertices and it has only $n+1$ edges, it would literally be a tree. So there is a vertex that is not incident to any deleted edges in $F$, let's remove this vertex. Now we need to remove 1 more vertex. Just remove any vertex in $H$. So now we have $n$ vertices left. There's a problem though, we might have removed more than 2 deleted edges from $F$ so now in the remaining graph, the number of deleted edges is less than $n-1$. Well, just arbitrarily remove more edges and make sure not to violate the condition on deleted edges. So now the remaining graph can use the IH to obtain a perfect matching. And note the edges used in this perfect matching are also present in the original graph (like the fact that we aribitrarily deleted some edges to satisfy the IH does not matter here). Also note the 2 vertices we deleted are adjacent to each other because one of the vertex is not incident to any of the deleted edges, so just add that edge to the matching and we have a perfect matching for the graph. □

## January 2017

**Question 1**

a) Solve the recurrence

$$\begin{cases} a_n = a_{n-1} + 5a_{n-2} + 3a_{n-3} & n \geq 3 \\ a_0 = 0, a_1 = 2, a_2 = 4 \end{cases}$$

*Proof.* The characteristic polynomial of the recurrence is $x^3 - x^2 - 5x - 3 = (x-3)(x+1)^2 = 0$ with roots $3, -1$ each of multiplicity 1 and 2. To factor the polynomials, one could use the rational root theorem to deduce that one of the roots of $\pm 3$ and use long division to obtain the other factor. Theorem 2.2.7 in the boook suggests that the solution looks like $a_n = A_1 3^n + A_2(-1)^n + A_3 n(-1)^n$. Solving the system of equation $A_1 + A_2 = a_0 = 1, 3A_1 + A_2 + A_3 = a_1 = 2, 9A_1 + A_2 + 3A_3 = a_2 = 4$, we obtain that $A_1 = \frac{9}{16}, A_2 = \frac{7}{16}, A_3 = -\frac{3}{4}$. So we get that

$$a_n = \frac{9}{16}3^n + \frac{7}{16}(-1)^n - \frac{3}{4}n(-1)^n$$

□

b) Prove that every polynomial satisfies some finite order, linear, homogeneous, constant coefficient recurrence.

*Proof.* WLOG let polynomial be $x^k - c_1 x^{k-1} - ... - c_k x^0$. The polynomial is monic because you can always divide the coefficient of the highest term if it was not. Now the recurrence $a_n = c_1 a_{n-1} + ... + c_k a_{n-k}$ is satisfied. □

**Question 8** Prove that if $G$ is an $X, Y-$bigraph with $|X| = |Y| = n$, then $\alpha'(G) \geq \min\{2\delta(G), n\}$. [Recall that $\alpha'(G)$ is the size of a largest matching in $G$.]

*Proof.* We proved in class that $\alpha' = \min_{S \subset X}\{|X| - \text{defect}(S)\}$. So we'll show $\min_{S \subset X}\{|X| - \text{defect}(S)\} \geq \min\{2\delta, n\}$ by showing for any $S \subset X$, $|X| - \text{defect}(S) \geq 2\delta$ or $|X| - \text{defect}(S) \geq n$. If $|X| - \text{defect}(S) \geq n$, then we're good. So suppose that $|X| - \text{defect}(S) < n$. Rearranging the inequality and substituting in $|X|$ and defect$(S)$, we get that $N(S) < |S|$. Note $S$ is not empty because if $S = \varnothing$, then $|X| - \text{defect}(S) = n \geq n$, a contradiction. So there is some vertex in $S$, so the neighborhood is at least $\delta$ big, i.e. $N(S) \geq \delta$. Note $Y \backslash N(S) \neq \varnothing$, because if it was emptyset, then $n \geq |S| > |N(S)| = |Y| = n$ is a contradiction. So there's some vertex in $Y \backslash N(S)$, this vertex must extend all of its $\geq \delta$ edges into $X \backslash S$ because if some of its edges went into $S$, it would have been in $N(S)$ instead of $Y \backslash N(S)$. So $|X \backslash S| = n - |S| \geq \delta$ as well. So we have

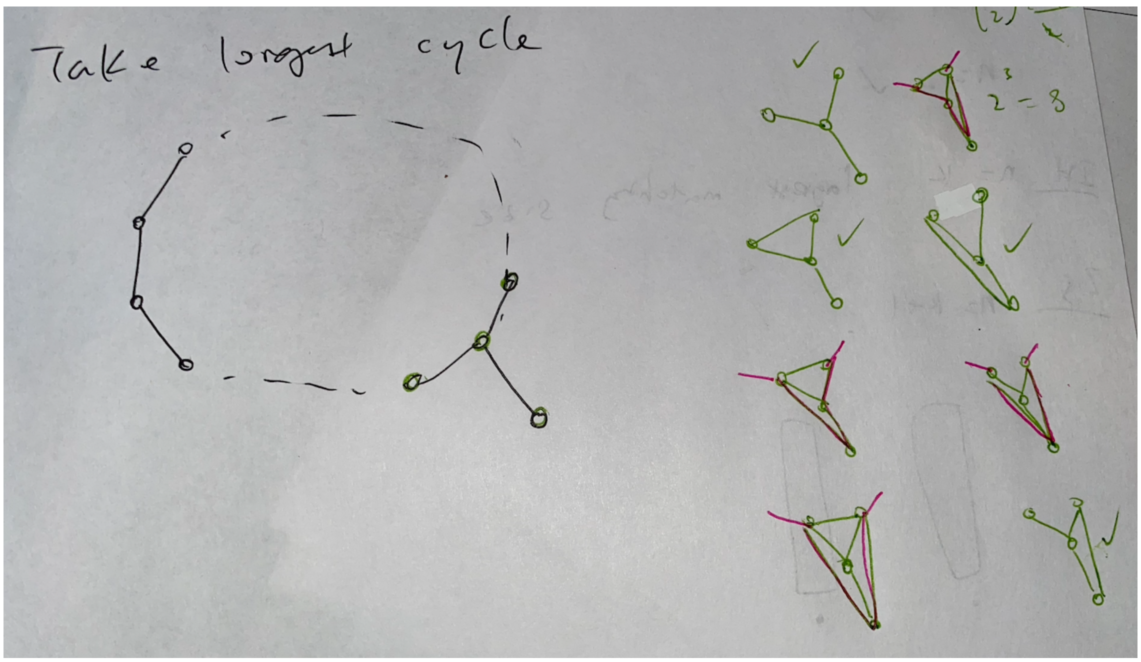
$$|X| - \text{defect}(S) = n - |S| + N(S) \geq 2\delta$$

as desired.

□

**Question 9** The graph below are called the claw and the paw respectively. Prove that a connected graph $G$ that contains a cycle and does not contain either a claw or a paw as an induced subgraph is Hamiltonion.

*Proof.* Take the longest cycle there is in $G$. AFSOC $G$ is not hamiltonian, this means that there is some vertex that did not get covered. Since $G$ is connected, it must be attached to the cycle, as shown in graph. Now we enumerate all the possible induced subgraphs of the four green vertices. Four of the possible induced subgraphs are claw or paw, so we found a contradiction. The remaining four we can augment the cycle by following the pink lines, contradicting that it's the longest cycle.




□